On Gauge Invariance and Symmetry of the Energy-Momentum Tensor

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Abstract

Noether's theorem links symmetries in physical systems to their respective conservation laws. In this paper, I explore Noether's theorem as it applies the the Lagrangian density for a free electromagnetic field, and use the variational approach - a standard textbook approach to derive the canonical energy-momentum tensor. However, the need for symmetry of the energy-momentum tensor requires additional steps like the Belinfante symmetrization procedure, which introduces spin angular momentum tensors. Yet, Bessel-Hagen's approach of taking advantage of gauge invariance addresses limitations, such as rotational contributions, of the canonical approach and derives a symmetric energy-momentum tensor.

1. Introduction

We begin with the premise of this derivation, Noether's theorem: asserting that for continuous symmetries of the action, there exists an associated conservation law [2]. Importantly, the action of a system is defined as the integral over its Lagrangian. In a standard application of Noether's theorem, we consider the Lagrangian density for the free electromagnetic field:

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}. \tag{1}$$

The version from L.D. Landau and E.M. Lifshitz takes the form of

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}, \tag{2}$$

but for the purpose of this report, I will stick to EQ 1, where $F_{\mu\nu}$ is the field tensor, defined as:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{3}$$

So as to not clutter the paper, I refer you to the appendix for properties and notations of tensors; for example, ∂^μ represents the differentiation with respect to the covariant spacetime variable x. Also, the four potential A^μ is a function of the spacetime variable x^μ, but for simplicity, I will often just use x in its place, however, the upper index μ will appear when it is relevant for the derivation.

Continuing, this derivation is particularly important because in it's canonical form, the energy-momentum tensor is not symmetric. However, the energy-momentum tensor must be symmetric to ensure consistency across different reference frames. For example, consider an inertial reference frame, despite moving ahead, we will see the symmetry of the form:

$$T^{\mu o} = T^{o\mu}, \tag{4}$$

which implies that in different reference frames, the energy and momentum are equivalent. And this makes physical sense because the flow of energy implies momentum, and any violation of this would make a physically unreasonable situation [3]. I also argue that from gauge invariance, as will be shown, the tensor is manifestly symmetric.

In order to achieve symmetrization, the tensor $T^{\mu\nu}$ is often subjected to a procedure proposed by Belifante. Indeed, it is strange that the fundamental nature of Noether's theorem does not produce a symmetric energy-momentum tensor, perhaps because the canonical tensor does not capture all the required symmetries. However, the approach by Besel-Hagen includes mixtures of different variations like gauge invariance where the variation approach for Noether's theorem is restricted - as we will see.

In order to show the Besel-Hagen approach using gauge invariance, it is necessary to work through the standard textbook approach and realize its shortcomings, thusly, addressing the shortcomings and where the Besel-Hagen approach succeeds. Additionally, while I include theory for the purpose derivation, this is more of a set of derivations following the approach from Helmut Haberzettl in his paper, *Using Gauge Invariance to Symmetrize the Energy-Momentum Tensor of Electrodynamics* [1].

2. Variational Approach

First, we begin with the textbook approach, the variational approach [4], where the action S is given by the integral of the Lagrangian density L over four-dimensional space:

$$S = \int \mathcal{L} d^4x. \tag{5}$$

We use the Lagrangian of a free electromagnetic field, which is a function of the four potential:

$$\mathcal{L} = \mathcal{L}(x^\mu, A^\nu, \partial^\mu A^\nu). \tag{6}$$

As with the usual variational procedures, we also consider fixed endpoints with small variations δA^μ and δx^μ:

$$x'^\mu = x^\mu + \delta x^\mu \ , \tag{7}$$

$$A'^\mu(x) = A^\mu(x) + \delta A^\mu(x). \tag{8}$$

By the principle of least action, the variation with fixed endpoints of the action is zero, $\delta S = 0$, which we derive from:

$$S = \int \mathcal{L}(x^\mu, A^\nu, \partial^\mu A^\nu) \ d^4x, \tag{9}$$

such that,

$$\begin{aligned}
\delta S &= \int \delta\mathcal{L}(x^\mu, A^\nu, \partial^\mu A^\nu)\, d^4x \\
&= \int \left(\frac{\partial \mathcal{L}}{\partial A^\nu}\delta A^\nu + \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}\delta(\partial^\mu A^\nu) + \partial^\mu \mathcal{L}\delta x^\mu \right) d^4x \\
&= \int \left(\frac{\partial \mathcal{L}}{\partial A^\nu}\delta A^\nu + \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}\partial^\mu(\delta A^\nu) + \partial^\mu \mathcal{L}\delta x^\mu \right) d^4x \\
&= \int \left(\frac{\partial \mathcal{L}}{\partial A^\nu}\delta A^\nu + \partial^\mu(\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}\delta A^\nu) - \partial^\mu(\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)})\delta A^\nu + \partial^\mu(\mathcal{L}\delta x^\mu) - \partial^\mu(\delta x^\mu)\mathcal{L} \right) d^4x.
\end{aligned} \tag{10}$$

In the last part, we take advantage of the variation of a constant from

$$\partial^\mu \partial x^\mu = \delta(\frac{\partial x^\mu}{\partial x^\mu}) = \delta(1) = 0. \tag{11}$$

Thus continuing the derivation of EQ 10:

$$\begin{aligned}
\delta S &= \int \left(\frac{\partial \mathcal{L}}{\partial A^\nu} - \partial^\mu(\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}) \right) \delta A^\nu d^4x + \int \partial^\mu \left(\mathcal{L}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}\delta A^\nu \right) d^4x \\
&= \int \left(\frac{\partial \mathcal{L}}{\partial A^\nu} - \partial^\mu(\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}) \right) \delta A^\nu d^4x + \int \left(\mathcal{L}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}\delta A^\nu \right) d\sigma^\mu = 0.
\end{aligned} \tag{12}$$

The integral in the second quantity becomes an integral over hyper surface through the following relation:

$$\partial^\mu d^4x = d\sigma^\mu. \tag{13}$$

From the Euler-Lagrange equations, the first integral becomes zero as:

$$\frac{\partial \mathcal{L}}{\partial A^\nu} - \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} \right) = 0. \tag{14}$$

Therefore, the second surface integral must also vanish:

$$\int \left(\mathcal{L}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} \delta A^\nu \right) d\sigma^\mu = 0. \tag{15}$$

The result from EQ 15 is the basis of Noether's theorem. First, we consider the variation in four-potential from EQ 8:

$$\begin{aligned} \delta A^\mu(x) &= A'^\mu(x) - A^\mu(x) \\ &= A'^\mu(x) - A^\mu(x) + A'^\mu(x') - A'^\mu(x') \\ &= [A'^\mu(x) - A'^\mu(x')] + [A'^\mu(x') - A^\mu(x)]. \end{aligned} \tag{16}$$

Following, we will take advantage of contravariant vector rules [5] such that:

$$A'^\alpha(x') = \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta(x). \tag{17}$$

So, looking at the second bracket in EQ 16, and since the primed potential is a function of x', then:

$$A'^\mu(x') - A^\mu(x) = \frac{\partial x'^\mu}{\partial x^\sigma} A^\sigma(x) - A^\mu(x). \tag{18}$$

We can further reduce from this term, as again using contravariant vector rules, and that the spacetime variable x is really x^μ, the partial component in EQ 18 becomes:

$$\begin{aligned} \frac{\partial}{\partial x^\sigma}(x'^\mu) &= \frac{\partial}{\partial x^\sigma}(x^\mu + \delta x^\mu) \\ &= \frac{\partial x^\mu}{\partial x^\sigma} + \partial_\sigma(\delta x^\mu) \\ &= \delta^\mu_\sigma + \partial_\sigma(\delta x^\mu). \end{aligned} \tag{19}$$

Accordingly, δx^μ is an infinitesimal variation in the space coordinate x^μ [6]. In taking its derivative, if the transformation is a translation, then the derivative will vanish, but if the Lorentz transformation is a rotation or boost, the derivative does not vanish. Here, the Haberzettl skips many steps, using definitional arguments. However, this is my take on how this formulation was derived, using the definition for rotations or boosts about an arbitrary axis, where the Lorentz transformation is defined as:

$$\begin{aligned} x'^\mu &= x^\mu + \delta x^\mu \\ &= \delta^\mu_\nu x^\nu + \omega^\mu{}_\nu x^\nu , \end{aligned} \tag{20}$$

Hence,

$$\begin{aligned} \frac{\partial}{\partial x^\sigma}(x'^\mu) &= \delta^\mu_\nu \delta^\nu_\sigma + \partial_\sigma(\omega^\mu{}_\nu x^\nu) \\ &= \delta^\mu_\sigma + \partial_\sigma(\omega^\mu{}_\nu x^\nu). \end{aligned} \tag{21}$$

We define the tensor $\omega^\mu{}_\nu$ as an antisymmetric tensor that contains rotation and boost parameters [7]. Therefore, I relate Haberzettl's transformation to the infinitesimal Lorentz transformation (from EQ 19 and EQ 21) as:

$$\partial_\sigma(\delta x^\mu) = \partial_\sigma(\omega^\mu{}_\nu x^\nu). \tag{22}$$

Thus, reworking the tensor [7], we can show that:

$$
\begin{aligned}
\partial_\sigma(\omega^\mu{}_\nu x^\nu) &= \partial_\sigma(g^{\mu\alpha}\omega_{\alpha\nu}x^\nu) \\
&= \partial_\sigma(g^{\mu\alpha}\delta^\beta_\nu\omega_{\alpha\beta}x^\nu) \\
&= \partial_\sigma\left(\frac{1}{2}\omega_{\alpha\beta}(g^{\mu\alpha}\delta^\beta_\nu - g^{\mu\beta}\delta^\alpha_\nu)x^\nu\right) \\
&= \partial_\sigma\left(\frac{1}{2}\omega_{\alpha\beta}(J^{\alpha\beta})^\mu{}_\nu x^\nu\right),
\end{aligned}
\tag{23}
$$

where $(J^{\alpha\beta})^\mu{}_\nu$ is a generator for boosts and rotations [8]. Now we can go back to EQ 22 and represent it as:

$$
\partial_\sigma(\delta x^\mu) = \delta^\mu_\sigma + \partial_\sigma\left(\frac{1}{2}\omega_{\alpha\beta}(J^{\alpha\beta})^\mu{}_\nu x^\nu\right). \tag{24}
$$

Continuing the derivation, plug in the result from EQ 24 into EQ 18, which gives,

$$
\begin{aligned}
A'^\mu(x') - A^\mu(x) &= [\delta^\mu_\sigma + \partial_\sigma\left(\frac{1}{2}\omega_{\alpha\beta}(J^{\alpha\beta})^\mu{}_\nu x^\nu\right)]A^\sigma(x) - A^\mu(x) \\
&= \delta^\mu_\sigma A^\sigma(x) + \frac{1}{2}\partial_\sigma\left(\omega_{\alpha\beta}(J^{\alpha\beta})^\mu{}_\nu x^\nu\right)A^\sigma(x) - A^\mu(x) \\
&= A^\mu(x) - A^\mu(x) + \frac{1}{2}\partial_\sigma\left(\omega_{\alpha\beta}(J^{\alpha\beta})^\mu{}_\nu x^\nu\right)A^\sigma(x) \\
&= \frac{1}{2}\partial_\sigma\left(\omega_{\alpha\beta}(J^{\alpha\beta})^\mu{}_\nu x^\nu\right)A^\sigma(x).
\end{aligned}
\tag{25}
$$

The derivative of $\omega_{\alpha\beta}$ vanishes so we can take it out of the derivative quantity from product rule, but the generator $(J^{\alpha\beta})^\mu{}_\nu$ does not vanish, so we can continue the derivation as by taking its derivative:

$$
\begin{aligned}
A'^\mu(x') - A^\mu(x) &= \frac{1}{2}\omega_{\alpha\beta}\partial_\sigma\left((J^{\alpha\beta})^\mu{}_\nu x^\nu\right)A^\sigma(x) \\
&= \frac{1}{2}\omega_{\alpha\beta}\partial_\sigma\left(g^{\mu\alpha}\delta^\beta_\nu x^\nu - g^{\mu\beta}\delta^\alpha_\nu x^\nu\right)A^\sigma(x) \\
&= \frac{1}{2}\omega_{\alpha\beta}\left(g^{\mu\alpha}\delta^\beta_\nu\delta^\nu_\sigma - g^{\mu\beta}\delta^\alpha_\nu\delta^\nu_\sigma\right)A^\sigma(x) \\
&= \frac{1}{2}\omega_{\alpha\beta}\left(g^{\mu\alpha}\delta^\beta_\sigma - g^{\mu\beta}\delta^\alpha_\sigma\right)A^\sigma(x) \\
&= \frac{1}{2}\omega_{\alpha\beta}\left(2g^{\mu\alpha}\delta^\beta_\sigma\right)A^\sigma(x) \\
&= \left(g^{\mu\alpha}\delta^\beta_\sigma\omega_{\alpha\beta}\right)A^\sigma(x) \\
&= (g^{\mu\alpha}\omega_{\alpha\sigma})A^\sigma(x) \\
&= (\omega^\mu{}_\sigma)A^\sigma(x) \\
&= \delta_1 A^\sigma(x).
\end{aligned}
\tag{26}
$$

Haberzettl denotes δ_1 as containing rotational variations, which agrees with my derivation. Now, EQ 16 can be rewritten as:

$$
\delta A^\mu(x) = [A'^\mu(x) - A'^\mu(x')] + \delta_1 A^\mu(x) \tag{27}
$$

where the index in the last term is swapped from σ to μ. We continue by only considering spacetime variations, such that the second term in the brackets - the potential as a function of x'- can be expanded by a first order Taylor expansion as:

$$
\begin{aligned}
A'^\mu(x'^\nu) &= A'^\mu(x^\nu + \delta x^\nu) \\
&= A'^\mu(x^\nu) + \delta x^\nu(\partial_\nu A^\mu(x)) \\
&= A'^\mu(x) + \delta x^\nu(\partial_\nu A^\mu(x)).
\end{aligned}
\tag{28}
$$

Therefore, the the whole bracketed quantity from EQ 27 becomes,

$$
\begin{aligned}
A'^\mu(x) - A'^\mu(x') &= A'^\mu(x) - [A'^\mu(x) + \delta x^\nu(\partial_\nu A^\mu(x))] \\
&= -\delta x^\nu(\partial_\nu A^\mu(x)).
\end{aligned}
\tag{29}
$$

Thus, we can further simply EQ 27 as:

$$\delta A^\mu(x) = -\delta x^\nu(\partial_\nu A^\mu(x)) + \delta_1 A^\mu(x). \tag{30}$$

But for the current approach, as per the Haberzettl, we ignored and will ignore rotational contributions, such that EQ 30 is left as,

$$\delta A^\mu(x) = -\delta x^\nu(\partial_\nu A^\mu(x)). \tag{31}$$

Finally, we can plug in the result from EQ 31, changing μ to ν and ν into σ - I believe that the author uses a method of the following form - into the integral from EQ 15:

$$\begin{aligned}
\int [\mathcal{L}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}\delta A^\nu] d\sigma^\mu &= \int [\mathcal{L}\delta x^\mu - \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}(\partial_\sigma A^\nu)\delta x^\sigma] d\sigma^\mu \\
&= \int [\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}(\partial_\sigma A^\nu)\delta x^\sigma - \mathcal{L}\delta x^\mu] d\sigma^\mu \\
&= \int [\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}(\partial_\sigma A^\nu) g^{\sigma\lambda}\delta x_\lambda - \mathcal{L} g^{\mu\lambda}\delta x_\mu] d\sigma^\mu \\
&= \int [\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} g^{\sigma\lambda}(\partial_\sigma A^\nu) - g^{\mu\lambda}\mathcal{L}]\delta x_\lambda d\sigma^\mu \\
&= \int [\frac{\partial \mathcal{L}}{\partial(g_{\mu\sigma}\partial_\sigma A^\nu)}(\partial^\lambda A^\nu) - g^{\mu\lambda}\mathcal{L}]\delta x_\lambda d\sigma^\mu \\
&= \int [g^{\mu\sigma}\frac{\partial \mathcal{L}}{\partial(\partial_\sigma A^\nu)}(\partial^\lambda A^\nu) - g^{\mu\lambda}\mathcal{L}]\delta x_\lambda d\sigma^\mu = 0.
\end{aligned} \tag{32}$$

The result of this derivation is the quantity in the integral, which is called the canonical energy-momentum tensor:

$$\begin{aligned}
T^{\mu\lambda} &= g^{\mu\sigma}\frac{\partial \mathcal{L}}{\partial(\partial_\sigma A^\nu)}(\partial^\lambda A^\nu) - g^{\mu\lambda}\mathcal{L} \\
&= -\frac{1}{\mu_0} g^{\mu\sigma} F_{\sigma\nu}(\partial^\lambda A^\nu) - g^{\mu\lambda}\mathcal{L} \\
&= -\frac{1}{\mu_0} F^\mu{}_\nu \partial^\lambda A^\nu - g^{\mu\lambda}\mathcal{L}.
\end{aligned} \tag{33}$$

But this is ultimately only valid for translational degrees of freedom since we dropped the rotational terms in the derivation, and it is not symmetric.

3. Effect of Neglected Rotational Contributions

Since we omitted the rotational term $\delta_1 A^\mu(x^\mu)$, we cannot derive any properties about rotation. To verify this, lets go back to EQ 32 and add the omitted term back into the equation, using EQ 30 instead of 31 for the variation in potential:

$$\begin{aligned}
\int [\mathcal{L}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}\delta A^\nu] d\sigma^\mu &= \int [\mathcal{L}\delta x^\mu - \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}\delta x^\sigma(\partial_\sigma A^\nu) + \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}\delta_1 A^\nu] d\sigma^\mu \\
&= \int [\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}\delta_1 A^\nu + \Big(\mathcal{L}\delta x^\mu - \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}\delta x^\sigma(\partial_\sigma A^\nu)\Big)] d\sigma^\mu \\
&= \int [\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}\delta_1 A^\nu + T^{\mu\lambda}\delta x_\lambda] d\sigma^\mu = 0.
\end{aligned} \tag{34}$$

Haberzettl defines the first term as the spin-angular momentum tensor. While he names it by definition, here is my take on the derivation using the results from EQ 25 and EQ 26:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial^{\mu} A^{\nu})}\delta_1 A^{\nu} &= -\frac{1}{\mu_0} F^{\mu\nu}\delta_1 A^{\nu} \\
&= -\frac{1}{\mu_0} F^{\mu\nu}[\frac{1}{2}\omega_{\alpha\beta}\partial_{\nu}\left((J^{\alpha\beta})^{\mu}{}_{\sigma} x^{\sigma}\right)]A^{\nu} \\
&= -\frac{1}{2\mu_0} F^{\mu\nu}\omega_{\alpha\beta}[\partial_{\nu}\left((J^{\alpha\beta})^{\mu}{}_{\sigma} x^{\sigma}\right)]A^{\nu} \\
&= -\frac{1}{2\mu_0} F^{\mu\nu}\omega_{\alpha\beta}[\partial_{\sigma}\left(g^{\mu\alpha}\delta^{\beta}_{\sigma} x^{\sigma} - g^{\mu\beta}\delta^{\alpha}_{\sigma} x^{\sigma}\right)]A^{\nu} \\
&= -\frac{1}{2\mu_0} F^{\mu\nu}\omega_{\alpha\beta}(g^{\mu\alpha}\delta^{\beta}_{\sigma}\delta^{\sigma}_{\nu} - g^{\mu\beta}\delta^{\alpha}_{\sigma}\delta^{\sigma}_{\nu})A^{\nu} \\
&= -\frac{1}{2\mu_0} F^{\mu\nu}\omega_{\alpha\beta}(g^{\mu\alpha}\delta^{\beta}_{\nu} - g^{\mu\beta}\delta^{\alpha}_{\nu})A^{\nu} \\
&= -\frac{1}{2\mu_0}\omega_{\alpha\beta}(g^{\mu\alpha}F^{\mu\nu}\delta^{\beta}_{\nu}A^{\nu} - g^{\mu\beta}F^{\mu\nu}\delta^{\alpha}_{\nu}A^{\nu}) \\
&= -\frac{1}{2\mu_0}\omega_{\alpha\beta}(g_{\mu\alpha}F^{\mu\nu}A^{\beta} - g_{\mu\beta}F^{\mu\nu}A^{\alpha}) \\
&= -\frac{1}{2\mu_0}\omega_{\alpha\beta}(F^{\nu}_{\alpha}A^{\beta} - F^{\nu}_{\beta}A^{\alpha})
\end{aligned} \tag{35}
$$

My derivation thus gets the form of the spin angular momentum tensor as:

$$
S^{\nu\beta}_{\alpha} = -\frac{1}{\mu_0}(F^{\nu}_{\alpha}A^{\beta} - F^{\nu}_{\beta}A^{\alpha}). \tag{36}
$$

While this results in an extra 1/2 term on $\omega_{\alpha\omega}$, it shows up later and get eliminated so I will keep it. Additionally, my form of the spin angular momentum can be transformed into Haberzettl's version by renaming α to ν and vice-versa, and raising quantities using the metric tensor:

$$
\begin{aligned}
S^{\mu\alpha\beta} &= g^{\mu\nu}S^{\alpha\beta}_{\nu} \\
&= -\frac{1}{\mu_0}(g^{\mu\nu}F^{\alpha}_{\nu}A^{\beta} - g^{\mu\nu}F^{\beta}_{\nu}A^{\alpha}) \\
&= -\frac{1}{\mu_0}(F^{\mu\alpha}A^{\beta} - F^{\mu\beta}A^{\alpha}).
\end{aligned} \tag{37}
$$

Hence, going back to EQ 34, we get,

$$
\int (\frac{1}{2}S^{\mu\alpha\beta}\omega_{\alpha\beta} - T^{\mu\lambda}\delta x_{\lambda})d\sigma^{\mu} = 0 \tag{38}
$$

Continuing with the procedure, taking advantage of the fact that $\partial_{\mu}F^{\mu\beta} = 0$ and $F^{\mu\alpha} = g_{\beta\alpha}F^{\mu\beta}$, we take the divergence of EQ 37:

$$
\begin{aligned}
\partial_{\mu}S^{\mu\alpha\beta} &= -\frac{1}{\mu_0}(F^{\mu\alpha}\partial_{\mu}A^{\beta} - F^{\mu\beta}\partial_{\mu}A^{\alpha}) \\
&= -\frac{1}{\mu_0}(g_{\beta\alpha}F^{\mu\beta}\partial_{\mu}A^{\beta} - F^{\mu\beta}\partial_{\mu}A^{\alpha}) \\
&= \frac{1}{\mu_0}F^{\mu\beta}(\partial_{\mu}g_{\beta\alpha}A^{\beta} - \partial_{\mu}A^{\alpha}) \\
&= \frac{1}{\mu_0}F^{\mu\beta}(\partial_{\mu}A_{\alpha} - \partial_{\mu}A^{\alpha}) \neq 0.
\end{aligned} \tag{39}
$$

Consistent with the results from Bliokh et. all [9], but does go to zero because the partial derivative is of different components, the contravariant and covariant versions of the four potential. Further:

$$
\int \partial_{\mu}[\left(\frac{1}{2}S^{\mu\alpha\beta}\omega_{\alpha\beta} - T^{\mu\lambda}(\frac{1}{2}\omega_{\alpha\beta}(J^{\alpha\beta})^{\mu}{}_{\lambda}x^{\lambda})\right)d\sigma^{\mu}] = \int \frac{1}{2}\omega_{\alpha\beta}\partial_{\mu}[S^{\mu\alpha\beta} - T^{\mu\lambda}\left((J^{\alpha\beta})^{\mu}{}_{\lambda}x^{\lambda}\right)]d^4x = 0. \tag{40}
$$

Since EQ 39 does not go to zero, and we derived it using the infinitesimal rotation term, this term would have to contribute to the Noether current for rotations, i.e., the flow carrying the charge. Therefore, considering the quantity in the integral in EQ 40, it must be zero:

$$\partial_\mu[S^{\mu\alpha\beta} - T^{\mu\lambda}\left((J^{\alpha\beta})^\mu{}_\lambda x^\lambda\right)] = 0. \tag{41}$$

Where we get a conservation law by adding rotational variation. This law must be satisfied, hence without adding the rotational variation term, we would only get the second term in the quantity, which would violate the law we just derived.

4. Belinfante Symmetrization

We proceed to the Belinfante procedure, a method for producing a symmetric energy-momentum tensor. The first step is to construct a tensor of three spin-angular momentum tensors, per the literature [9],

$$K^{\mu\sigma\lambda} = -\frac{1}{2}(S^{\mu\sigma\lambda} + S^{\nu\sigma\lambda} - S^{\sigma\mu\lambda}). \tag{42}$$

Second, we produce a new tensor by adding the divergence of tensor K to the canonical energy momentum tensor:

$$T'^{\mu\lambda} = T^{\mu\lambda} + \partial_\sigma K^{\mu\sigma\lambda}. \tag{43}$$

Then, in taking the divergence of the new tensor of the newly constructed tensor $T'^{\mu\lambda}$,

$$\begin{aligned} \partial_\mu T'^{\mu\lambda} &= \partial_\mu T^{\mu\lambda} + \partial_\mu\partial_\sigma K^{\mu\sigma\lambda} \\ &= \partial_\mu T^{\mu\lambda}. \end{aligned} \tag{44}$$

where the second partial derivative vanishes as K is antisymmetric, and the double contraction always vanishes - consistent with textbook results [9]. Thus, using results from EQ 37 and 38, we can evaluate tensor K as:

$$\begin{aligned} K^{\mu\sigma\lambda} &= -\frac{1}{2}(S^{\mu\sigma\lambda} + S^{\lambda\sigma\mu} - S^{\sigma\mu\lambda}) \\ &= -\frac{1}{2}[-\frac{1}{\mu_0}(F^{\mu\sigma}A^\lambda - F^{\mu\lambda}A^\sigma) - \frac{1}{\mu_0}(F^{\lambda\sigma}A^\mu - F^{\lambda\mu}A^\sigma) + \frac{1}{\mu_0}(F^{\sigma\mu}A^\lambda - F^{\sigma\lambda}A^\mu)] \\ &= \frac{1}{2\mu_0}[(F^{\mu\sigma} - F^{\sigma\mu})A^\lambda + (F^{\lambda\sigma} + F^{\sigma\lambda})A^\mu + (F^{\mu\lambda} - F^{\lambda\mu})A^\sigma] \\ &= \frac{1}{2\mu_0}(2F^{\mu\sigma}A^\lambda) \\ &= \frac{1}{\mu_0}F^{\mu\sigma}A^\lambda, \end{aligned} \tag{45}$$

taking advantage of the field tensor,

$$F^{\mu\sigma} = -F^{\sigma\mu}, \tag{46}$$

and thus:

$$\begin{aligned} \partial_\sigma K^{\mu\sigma\lambda} &= \frac{1}{\mu_0}\partial_\sigma F^{\mu\sigma}A^\lambda + \frac{1}{\mu_0}F^{\mu\sigma}\partial_\sigma A^\lambda \\ &= \frac{1}{\mu_0}F^{\mu\sigma}\partial_\sigma A^\lambda. \end{aligned} \tag{47}$$

Adding the result from EQ 47 to EQ 43 and recalling canonical energy-momentum tensor in EQ 33, we can write:

$$
\begin{aligned}
T'^{\mu\lambda} &= T^{\mu\lambda} + \frac{1}{\mu_0} F^{\mu\sigma}\partial_\sigma A^\lambda \\
&= -\frac{1}{\mu_0} F^{\mu}{}_{\nu}\partial^\lambda A^\nu - g^{\mu\lambda}\mathcal{L} + \frac{1}{\mu_0} F^{\mu\sigma}\partial_\sigma A^\lambda \\
&= -\frac{1}{\mu_0} F^{\mu}{}_{\nu}\partial^\lambda A^\nu - g^{\mu\lambda}\mathcal{L} + \frac{1}{\mu_0} g^{\sigma\nu} F^{\mu}{}_{\nu}\partial_\sigma A^\lambda \\
&= -\frac{1}{\mu_0} F^{\mu}{}_{\nu}\partial^\lambda A^\nu - g^{\mu\lambda}\mathcal{L} + \frac{1}{\mu_0} F^{\mu}{}_{\nu}\partial^\nu A^\lambda \\
&= \frac{1}{\mu_0} F^{\mu}{}_{\nu}(\partial^\nu A^\lambda + \partial^\lambda A^\nu) - g^{\mu\lambda}\mathcal{L} \\
&= \frac{1}{\mu_0} F^{\mu}{}_{\nu} F^{\nu\lambda} - g^{\mu\lambda}\mathcal{L}.
\end{aligned} \tag{48}
$$

This new tensor is indeed symmetric. And just following procedure of Noether's theorem, from EQ 41 and of the same formulation as of EQ 40 but plugging in the new primed tensor, we still get a conservation law in the form of:

$$
\partial_\mu [S^{\mu\alpha\beta} - T^{\mu\lambda}\left((J^{\alpha\beta})^{\mu}{}_{\lambda} x^\lambda\right) + \partial_\sigma K^{\mu\sigma\lambda}\left((J^{\alpha\beta})^{\mu}{}_{\lambda} x^\lambda\right)] = 0. \tag{49}
$$

5. Gauge Invariance

Up to this point, the derivations relied on the first order Taylor series expansion of the spacetime coordinate δx^μ. Nevertheless, as Bessel-Hagen highlight, the Noether framework offers a way to incorporate symmetries beyond just those of spacetime [10]. Notably, we consider gauge invariance in electrodynamics, where the standard four potential under gauge transformation take the form:

$$
A'^\nu(x) = A^\nu(x) - \partial^\nu\phi(x). \tag{50}
$$

Here, ϕ is a scalar function. Beginning the procedure, we first recall EQ 8, the transformation for a small variation, then Haberzettl anticipates that that variation will be split into two terms, a variation term in the spacetime coordinate and a variation term due to gauge transformation:

$$
\delta A^\mu(x) = \delta_x A^\mu(x) - \delta_g A^\mu(x). \tag{51}
$$

Now we employ the gauge transformation from EQ 50 into EQ 27 from the variational approach:

$$
\begin{aligned}
\delta A^\mu(x) &= [A'^\mu(x) - A'^\mu(x')] + \delta_1 A^\mu(x) \\
&= [A^\mu(x) - \partial^\mu\phi(x) - A^\mu(x') + \partial^\mu\phi(x')] + \delta_1 A^\mu(x) \\
&= (A^\mu(x) - A^\mu(x')) + (\partial^\mu\phi(x') - \partial^\mu\phi(x)) + \delta_1 A^\mu(x) \\
&= (A^\mu(x) - A^\mu(x')) + (\partial^\mu\phi(x') - \partial^\mu\phi(x)) + \delta_1 A^\mu(x) \\
&= (A^\mu(x) - A^\mu(x')) + \partial^\mu\left(\phi(x') - \phi(x)\right) + \delta_1 A^\mu(x) \\
&= (A^\mu(x) - A^\mu(x')) + \partial^\mu\delta\phi + \delta_1 A^\mu(x).
\end{aligned} \tag{52}
$$

Here, Haberzettl denotes

$$
\delta\phi(x) = \phi(x') - \phi(x). \tag{53}
$$

And again, for the term in the bracket, by a first order Taylor expansion, we get:

$$
\begin{aligned}
A^\mu(x) - A^\mu(x') &= A^\mu(x) - A^\mu(x+\delta x) \\
&= A^\mu(x) - A^\mu(x) - \delta x^\nu(\partial^\nu A^\mu) \\
&= -\delta x^\nu(\partial_\nu A^\mu),
\end{aligned} \tag{54}
$$

similar to the result from the initial approach. Therefore EQ 52 becomes:

$$
\delta A^\mu(x) = -\delta x^\nu(\partial_\nu A^\mu) + \partial^\mu\delta\phi(x) + \delta_1 A^\mu(x), \tag{55}
$$

giving the following equivalence:

$$
\begin{aligned}
\delta A^{\mu}(x) &= \delta_x A^{\mu}(x) - \delta_g A^{\mu}(x) \\
&= -\delta x^{\nu}(\partial_{\nu} A^{\mu}(x)) + \partial^{\mu}\delta\phi + \delta_1 A^{\mu}(x).
\end{aligned} \tag{56}
$$

Thus, the spacetime variation is the exact result from the original approach in section 2:

$$
\delta_x A^{\mu}(x) = -\delta x^{\nu}(\partial_{\nu} A^{\mu}), \tag{57}
$$

and the remaining term accounts for gauge invariance as:

$$
\delta_g A^{\mu}(x) = \partial^{\mu}\delta\phi + \delta_1 A^{\mu}(x). \tag{58}
$$

Further, the variation infinitesimal of the scalar,

$$
\begin{aligned}
\delta\phi &= \phi(x') - \phi(x) \\
&= \phi(x^{\nu} - \delta x^{\nu}) - \phi(x^{\nu}) \\
&= \phi(x^{\nu}) + \delta x^{\nu}(\partial_{\nu}\phi) - \phi(x^{\nu}) \\
&= \delta x^{\nu}(\partial_{\nu}\phi) \\
&= \delta x^{\nu}(A_{\nu}(x) - A'_{\nu}(x)).
\end{aligned} \tag{59}
$$

But, according to Haberzettl, we only use the scalar form linear in the field, which eliminates the primed potential and leaves,

$$
\delta\phi = \delta x^{\nu} A_{\nu}(x). \tag{60}
$$

Plugging the result from EQ 60 into the gauge invariance variation, EQ 58, then:

$$
\begin{aligned}
\delta_g A^{\mu}(x) &= \partial^{\mu}\left(\delta x^{\nu} A_{\nu}(x)\right) + \delta_1 A^{\mu}(x) \\
&= \delta x^{\nu}\partial^{\mu} A_{\nu}(x) + A_{\nu}(x)\partial^{\mu}\delta x^{\nu} + \delta_1 A^{\mu}(x).
\end{aligned} \tag{61}
$$

Again, taking advantage of EQ 22 and using the exact same procedure just with different indexes as in EQ 24 and EQ 26, my take on the second term from EQ 61 goes as the following:

$$
\begin{aligned}
A_{\nu}\partial^{\mu}\delta x^{\nu} &= A_{\nu}\frac{1}{2}\omega_{\alpha\beta}\partial^{\mu}\left((J^{\alpha\beta})^{\lambda}{}_{\sigma} x^{\sigma}\right) \\
&= A_{\nu}\frac{1}{2}\omega_{\alpha\beta} g^{\mu\nu}\partial_{\nu}\left((J^{\alpha\beta})^{\lambda}{}_{\sigma} x^{\sigma}\right) \\
&= \frac{1}{2}\omega_{\alpha\beta}\partial_{\nu}\left(g^{\lambda\alpha}\delta^{\beta}_{\sigma} x^{\sigma} - g^{\lambda\beta}\delta^{\alpha}_{\sigma} x^{\sigma}\right) A^{\mu} \\
&= \frac{1}{2}\omega_{\alpha\beta}\left(g^{\lambda\alpha}\delta^{\beta}_{\sigma}\delta^{\sigma}_{\nu} - g^{\lambda\beta}\delta^{\alpha}_{\sigma}\delta^{\sigma}_{\nu}\right) A^{\mu} \\
&= \frac{1}{2}\omega_{\alpha\beta}\left(g^{\lambda\alpha}\delta^{\beta}_{\nu} - g^{\lambda\beta}\delta^{\alpha}_{\nu}\right) A^{\mu} \\
&= \frac{1}{2}\omega_{\alpha\beta}\left(2g^{\lambda\alpha}\delta^{\beta}_{\nu}\right) A^{\mu} \\
&= (g^{\lambda\alpha}\delta^{\beta}_{\nu}\omega_{\alpha\beta})A^{\mu} \\
&= (g^{\lambda\alpha}\omega_{\alpha\nu})A^{\mu} \\
&= (\omega^{\lambda}{}_{\nu})A^{\mu} \\
&= \delta_2 A^{\mu}(x).
\end{aligned} \tag{62}
$$

Where again, Haberzettl denotes δ_2 as rotational contributions of the same form as δ_1, which agrees with my derivation. Thus, EQ 61 turns into:

$$
\delta_g A^{\mu}(x) = \delta x^{\nu}\partial^{\mu} A_{\nu}(x) + \delta_2 A^{\mu}(x) + \delta_1 A^{\mu}(x). \tag{63}
$$

Only considering the second two rotational terms:

$$
\begin{aligned}
\delta_1 A^{\mu}(x) + \delta_2 A^{\mu}(x) &= (\omega^{\nu}{}_{\nu})A^{\mu} + (\omega^{\lambda}{}_{\nu})A^{\mu} \\
&= \frac{1}{2}\omega_{\alpha\beta}\partial_{\sigma}\left((J^{\alpha\beta})^{\mu}{}_{\nu} x^{\nu}\right) A^{\sigma} + \frac{1}{2}\omega_{\alpha\beta}\partial^{\mu}\left((J^{\alpha\beta})^{\lambda}{}_{\sigma} x^{\sigma}\right) A_{\nu} \\
&= \frac{1}{2}\omega_{\alpha\beta}[A^{\sigma}\partial_{\sigma}\left((J^{\alpha\beta})^{\mu}{}_{\nu} x^{\nu}\right) + A_{\nu}\partial^{\mu}\left((J^{\alpha\beta})^{\lambda}{}_{\sigma} x^{\sigma}\right)] = 0
\end{aligned} \tag{64}
$$

Again, I refer to another arguement from Haberzettl, that the rotational terms will cancel out. Perhaps, this is because J is an antisymmetric tensor, where taking different partial derivatives of the same antisymmetric tensor cancels out. Therefore, we are left with:

$$\delta_g A^\mu(x) = \delta x^\nu \partial^\mu A_\nu(x). \tag{65}$$

Thus going back to EQ 56, we are left with:

$$\begin{aligned}
\delta A^\mu(x) &= \delta_x A^\mu(x) - \delta_g A^\mu(x) \\
&= -\delta x^\nu(\partial_\nu A^\mu) + \delta x^\nu \partial^\mu A_\nu(x) \\
&= -(\partial_\nu A^\mu - \partial^\mu A_\nu)\delta x^\nu \\
&= -F^\mu{}_\nu \delta x^\nu
\end{aligned} \tag{66}$$

Employing this variation back into EQ 15, but switching the μ with ν and replacing ν with σ,

$$\begin{aligned}
\int [\mathcal{L}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}\delta A^\nu] d\sigma^\mu &= \int [\mathcal{L}\delta x^\mu - \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} F^\nu_\sigma \delta x^\sigma] d\sigma^\mu \\
&= \int [\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} F^\nu_\sigma \delta x^\sigma - \mathcal{L}\delta x^\mu] d\sigma^\mu \\
&= \int [\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} F^\nu_\sigma g^{\sigma\lambda}\delta x_\lambda - g^{\mu\lambda}\delta x_\lambda \mathcal{L}] d\sigma^\mu \\
&= \int [\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} g^{\sigma\lambda} F^\nu_\sigma - g^{\mu\lambda}\mathcal{L}]\delta x_\lambda d\sigma^\mu \\
&= \int [\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} F^{\nu\lambda} - g^{\mu\lambda}\mathcal{L}]\delta x_\lambda d\sigma^\mu \\
&= \int [\frac{\partial \mathcal{L}}{\partial(g_{\mu\sigma}\partial_\sigma A^\nu)} F^{\nu\lambda} - g^{\mu\lambda}\mathcal{L}]\delta x_\lambda d\sigma^\mu \\
&= \int [g^{\mu\sigma}\frac{\partial \mathcal{L}}{\partial(\partial_\sigma A^\nu)} F^{\nu\lambda} - g^{\mu\lambda}\mathcal{L}]\delta x_\lambda d\sigma^\mu = 0,
\end{aligned} \tag{67}$$

where the result inside of the tensor gives the following tensor:

$$\begin{aligned}
T^{\mu\lambda} &= g^{\mu\sigma}\frac{\partial \mathcal{L}}{\partial(\partial_\sigma A^\nu)} F^{\nu\lambda} - g^{\mu\lambda}\mathcal{L} \\
&= -\frac{1}{\mu_0} g^{\mu\sigma} F_{\sigma\nu} F^{\nu\lambda} - g^{\mu\lambda}\mathcal{L} \\
&= -\frac{1}{\mu_0} F^\mu_\nu F^{\nu\lambda} - g^{\mu\lambda}\mathcal{L} \\
&= -\frac{1}{\mu_0} g_{\nu\sigma} F^{\sigma\mu} F^{\nu\lambda} - g^{\mu\lambda}\mathcal{L} \\
&= \frac{1}{\mu_0} g_{\nu\sigma} F^{\sigma\mu} F^{\lambda\nu} - g^{\mu\lambda}\mathcal{L}.
\end{aligned} \tag{68}$$

Hence, the new version of $T^{\mu\lambda}$ is ultimately, a symmetric tensor, derived via the variational approach, but taking additional steps of accounting for gauge invariant quantities.

6. Conclusion

The presentation in this paper, followed Haberzettl's exploration of Noether's theorem for a the Lagrangian for a free electromagnetic field. By employing the variational approach, we derived the canonical energy-momentum tensor, where its anti-symmetry poses challenges by not including rotational contributions.

To address this, we explored the Belinfante procedure, a textbook procedure use to symmetrize the energy-momentum tensor by introducing a combination of spin angular momentum tensors. However, Bessel-Hagen's approach is much more intuitive, where symmetry of the energy-momentum tensor is just a manifest property of the variational approach and addresses limitations like rotational contributions.

Ultimately, this is a set of derivations based off of the work from Haberzettl, hence, for a more theory grounded approach, refer to his original paper [1].

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A. Appendix: Relevant Equations

Throughout the derivations, I take advantage of many of the following properties of tensors, metric tensors, and contravariant and covariant vector rules, so as to not clutter the paper, these are taken from various sources mentioned in the references and relevant sections.

The metric tensor, as a symmetric tensor, satisfies:

$$g_{\alpha\beta} = g_{\beta\alpha}.$$

Thus, in contravariant and in mixed form:

$$g_{\alpha\beta} = g_{\alpha}^{\beta} = g^{\alpha\beta}.$$

Following, any two metric tensors produce the following:

$$g_{\alpha\gamma}g^{\gamma\beta} = \delta_{\alpha}^{\beta} = \delta_{\alpha\beta} = \delta^{\alpha\beta}.$$

A scalar in contravariant or covariant form can be transformed into the other by metric multiplication:

$$x_{\alpha} = g_{\alpha\beta}x^{\alpha}$$
$$x^{\alpha} = g^{\alpha\beta}x_{\alpha}$$

We denote differential with respect to covariant or contravariant vectors as:

$$\partial^{\alpha} = \frac{\partial}{\partial x_{\alpha}},$$

$$\partial_{\alpha} = \frac{\partial}{\partial x^{\alpha}}.$$

Hence, differentiation with respect to covariant or contravariant vector transform like:

$$\frac{\partial}{\partial x_{\alpha}} = g_{\alpha\beta}\frac{\partial}{\partial x^{\alpha}},$$

$$\partial^{\alpha} = g_{\alpha\beta}\partial_{\alpha}.$$

On the topic of derivation, the derivative of the spacetime variable with respect to another:

$$\frac{\partial}{\partial x^{\alpha}}x^{\beta} = \delta^{\beta\alpha}$$

Similarly, we can transform tensors with the metric tensor as:

$$A^{\mu\nu} = g^{\mu\lambda}g^{\nu\sigma}A_{\lambda}\sigma,$$

$$A_{\mu\nu} = g_{\mu\lambda}g_{\nu\sigma}A^{\lambda\sigma},$$

$$A^{\mu\nu} = g^{\mu\lambda}A^{\nu\lambda},$$

$$A_{\mu\nu} = g_{\mu\lambda}A_{\nu}^{\lambda},$$

$$A_{\nu}^{\lambda} = g^{\lambda\mu}A_{\mu\nu}.$$